Stable de Sitter critical points of the cosmology in quadratic gravitation with torsion

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Homogeneous isotropic spatial flat cosmological models with two torsion functions in vacuum are built and investigated in the framework of de Sitter gauge theory of gravity. It is shown that by certain choices of parameters of gravitational Lagrangian the cosmological equations have some exact constant solutions that turn out to be stable de Sitter critical points of dynamical systems and can explain observable acceleration of cosmological expansion. The role of the space-time torsion provoking the acceleration of cosmological expansion is shown.

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I. INTRODUCTION

Recently there has been a burst of activity dealing with quadratic gravitation. For example, the curvature-squared terms added to the usual Einstein action with cosmological constant have played a role in two recent investigations of four-dimensional gravity: in critical gravity [1], and in a pure Weyl-squared action considered by Maldacena [2].

The critical gravity provides a consistent toy model for quantum gravity as a useful simplified arena for studying some aspects of a potentially renormalisable theory of massless spin-2 fields in four dimensions.

The conformal gravity theory has been advanced as a candidate alternative to standard Einstein gravity. As a quantum theory the conformal theory is both renormalizable and unitary, with unitarity being obtained because the theory is a PT symmetric rather than a Hermitian theory. Because the variation of the conformal action leads to fourth-order equations of motion, it had long been thought that the theory would not be unitary. However, as has been shown by Bender and Mannheim [3] that one can find a realization of the theory that is unitary. Consequently, conformal gravity is to be regarded as a bona fide quantum gravitational theory. The conformal gravity theory can quite naturally handle some of the most troublesome problems in physics, the quantum gravity problem, the vacuum energy problem, and the dark matter problem. [4]

As a modified gravity theory quadratic gravitation has been used in cosmology [5]. In order to explain observable acceleration of cosmological expansion some authors introduce torsion terms in quadratic gravitation [6]. The quantum aspects of torsion theory and the possibility of the space-time torsion to exist and to be detected have been discussed in [7].The astronomical observations show that our universe is probably an asymptotically de Sitter (dS) one with a positive cosmological constant Λ [8]. If a gravitational

theory of Yang-Mills type is constructed starting from de Sitter gauge invariance principle, its gravitational Lagrangian naturally turns out to be the one of quadratic gravitation with torsion as will be shown in this paper. Therefore, a investigation of quadratic gravitation with torsion and its cosmological solutions expressed by de Sitter critic points will be carried out. The field equations will be derived. These equations are quite different from the equations obtained from Riemannian geometry based quadratic Lagrangians when varied with respect to the metric. Applying to the space flat FRW cosmology some de Sitter critical point solutions will be obtained. The stability of them will be analyzed.

The paper is organized as follows. In section II, starting from a Clifford algebra $C(3,1)$ the gravitational Lagrangian of a de Sitter gauge theory is constructed, the Lagrange equations of gravitational fields are derived. Applying them to a spatial flat universe the cosmological equations are obtained in section III. The vacuum solutions of these equations in two specific cases are presented in section IV. These two models correspond to the conformal cosmology of Mannheim [4,9] and the zero-energy gravity of Deser and Tekin [10], respectively. In contrast to them, the tetrad and the spin connection are taken to be the basic field variables and the torsion plies a important role here. In these specific models the cosmological equations are written as some dynamical systems, the real de Sitter critical points of them are obtained. Among these points, the stable ones which turn out to be exact constant solutions and describe the asymptotic behavior of the universe are found. In section V some concluding remarks are given. In Appendixes the calculations for stability analysis are presented.

II. LAGRANGIAN AND FIELD EQUATIONS

We begin with a brief introduction of a de Sitter gauge theory. In a gravitational gauge theory coupled to matter sources involving Dirac fields it is convenient to take Dirac matrices γ_I and their commutators $\sigma_{IJ} = \frac{1}{2}[\gamma_I, \gamma_J]$ as the basis of the gauge algebra. In this case we are led to a de Sitter gauge theory. Let $\{\gamma_I\}$ $(I = 0, 1, 2, 3)$ be a basis of an inner product space with signature $(-, +, +, +)$. A Clifford algebra $C(3,1)$ can be constructed by introducing the condition

$$\gamma_I\gamma_J + \gamma_J\gamma_I = 2\eta_{IJ} I. \tag{1}$$

with η_{IJ} =diag$(-1; 1; 1; 1)$. There is a 10-dimensional subspace of $C(3,1)$ which is a Lie algebra with basis $\gamma_5\gamma_I$ and $\sigma_{IJ} = \frac{1}{2}[\gamma_I, \gamma_J]$. This is the Lie algebra of a de Sitter group. We can introduce a connection [11,12]

$$\omega = \Gamma + \frac{1}{l}\gamma_5 \mathbf{e}, \tag{2}$$

defined by

$$\mathbf{e} = e^I{}_\mu \gamma_I \otimes dx^\mu, \tag{3}$$

and

$$\Gamma = \frac{1}{4}\Gamma^{IJ}{}_\mu \sigma_{IJ} \otimes dx^\mu,$$

where l denotes a constant with the dimension of length. The curvature of ω is

$$\Omega = d\omega + \frac{1}{2}[\omega, \omega] = \mathbf{R} + \frac{1}{l}\gamma_5 \mathbf{T} - \frac{1}{l^2}\mathbf{V}, \tag{4}$$

where

$$\begin{aligned} \mathbf{R} &= d\Gamma + \frac{1}{2}[\Gamma, \Gamma], \\ \mathbf{T} &= d\mathbf{e} + [\Gamma, \mathbf{e}], \\ \mathbf{V} &= \frac{1}{2}[\mathbf{e}, \mathbf{e}]. \end{aligned} \tag{5}$$

The Lorentz curvature $\mathbf{R}$, the torsion $\mathbf{T}$, and the cosmological term $\mathbf{V}$ are given by, respectively,

$$\begin{aligned} \mathbf{R} &= \frac{1}{8}R^{IJ}{}_{\mu\nu}\sigma_{IJ} \otimes dx^\mu \wedge dx^\nu, \\ \mathbf{T} &= \frac{1}{2}T^I{}_{\mu\nu}\sigma_{IJ} \otimes dx^\mu \wedge dx^\nu, \\ \mathbf{V} &= e^I{}_\mu e^J{}_\nu \sigma_{IJ} \otimes dx^\mu \wedge dx^\nu, \end{aligned} \tag{6}$$

with

$$R^{IJ}{}_{\mu\nu} = \partial_\mu \Gamma^{IJ}{}_\nu - \partial_\nu \Gamma^{IJ}{}_\mu + \eta_{KL}\Gamma^{IK}{}_\mu \Gamma^{LJ}{}_\nu - \eta_{KL}\Gamma^{IK}{}_\nu \Gamma^{LJ}{}_\mu, \tag{7}$$

and

$$T^I{}_{\mu\nu} = \partial_\mu e^I{}_\nu - \partial_\nu e^I{}_\mu + \Gamma^I{}_{J\mu} e^J{}_\nu - \Gamma^I{}_{J\nu} e^J{}_\mu. \tag{8}$$

Based on the local gauge invariance principle the gravitational Lagrangian can be made up of a quadratic term of the curvature Ω and its Hodge dual $*\Omega$:

$$\mathcal{L} = -\frac{1}{8}Tr\,(*\Omega \wedge \Omega) = \left(\frac{1}{32}R_{\mu\nu}{}^{\rho\sigma}R^{\mu\nu}{}_{\rho\sigma} - \frac{1}{4}l^{-2}T^\mu{}_{\nu\rho}T_\mu{}^{\nu\rho} + \frac{1}{2}l^{-2}R - 12l^{-4}\right)e, \tag{9}$$

where

$$e = \det\left|e^I{}_\mu\right|. \tag{10}$$

In four dimensional spacetime the Gauss-Bonnet term $\sqrt{-g}\left[R_{\mu\nu\lambda\tau}R^{\mu\nu\lambda\tau}-4R_{\mu\nu}R^{\mu\nu}+R^2\right]$ is purely topological and then the Lagrangian can be taken as

$$\mathcal{L}=-\frac{1}{8}Tr\left(*\Omega\wedge\Omega\right)=\left(\frac{1}{8}R_{\mu\nu}R^{\mu\nu}-\frac{1}{32}R^2-\frac{1}{4}l^{-2}T^{\mu}{}_{\nu\rho}T_{\mu}{}^{\nu\rho}+\frac{1}{2}l^{-2}R-12l^{-4}\right)e. \tag{11}$$

For the sake of a neater argument we extend the Lagrangian to including the coefficients

$$\beta=\frac{1}{8}l^2,\alpha=-\frac{1}{32}l^2,\gamma=-\frac{1}{4}, \tag{12}$$

and rewrite (10) as

$$\mathcal{L}=\left(\beta l^{-2}R_{\mu\nu}R^{\mu\nu}+\alpha l^{-2}R^2+\gamma l^{-2}T^{\mu}{}_{\nu\rho}T_{\mu}{}^{\nu\rho}+\frac{1}{2}l^{-2}R-12l^{-4}\right)e=Le, \tag{13}$$

with

$$L=\beta l^{-2}R_{\mu\nu}R^{\mu\nu}+\alpha l^{-2}R^2+\gamma l^{-2}T^{\mu}{}_{\nu\rho}T_{\mu}{}^{\nu\rho}+\frac{1}{2}l^{-2}R-12l^{-4}. \tag{14}$$

$\mathcal{L}$ is just the Lagrangian of quadratic-curvature gravities [10] with torsion.

The variational principle yields the field equations for the tetrad $e_I{}^{\mu}$ and the spin connection $\Gamma^{IJ}{}_{\mu}$

$$\begin{aligned}\frac{\delta\mathcal{L}}{\delta e_I{}^{\mu}} &= eE^I{}_{\mu},\\ \frac{\delta\mathcal{L}}{\delta\Gamma^{IJ}{}_{\mu}} &= es_{IJ}{}^{\mu},\end{aligned} \tag{15}$$

where $E^I{}_{\mu}$ and $s_{IJ}{}^{\mu}$ are energy- momentum and spin tensors of the matter source, respectively, the variational derivatives are given by

$$\begin{aligned}&\frac{\delta\mathcal{L}}{\delta e_I{}^{\mu}}\\ =\ &\{\beta l^{-2}\left(2e^{I\sigma}R^{\rho}{}_{\sigma}R_{\rho\mu}+2e^J{}_{\rho}R^{\rho\sigma}R^I{}_{J\mu\sigma}-e^I{}_{\mu}R_{\rho\sigma}R^{\rho\sigma}\right)+\alpha l^{-2}\left(4e^{I\nu}R_{\nu\mu}-e^I{}_{\mu}R\right)R\\ &+\gamma l^{-2}\left(4e^{I\nu}T^{\lambda}{}_{\nu\tau}T_{\lambda\mu}{}^{\tau}-4\partial_{\nu}\left(e^{I\lambda}T_{\mu\lambda}{}^{\nu}\right)-e^I{}_{\mu}T^{\lambda}{}_{\rho\sigma}T_{\lambda}{}^{\rho\sigma}+\left(4e^{I\lambda}T_{\mu\lambda}{}^{\nu}\right)e^K{}_{\tau}\partial_{\nu}e_K{}^{\tau}\right)\\ &+l^{-2}\left(e^{I\nu}R_{\nu\mu}-\frac{1}{2}e^I{}_{\mu}R\right)+12l^{-4}e^I{}_{\mu}\}e,\end{aligned} \tag{16}$$

$$\begin{aligned}\frac{\delta\mathcal{L}}{\delta\Gamma^{IJ}{}_{\mu}} =\ &\{2\beta l^{-2}e_J{}^{\lambda}[e_I{}^{\mu}\partial_{\nu}R_{\lambda}{}^{\nu}-e_I{}^{\nu}\partial_{\nu}R_{\lambda}{}^{\mu}+\left(e_I{}^{\nu}R_{\lambda}{}^{\mu}-e_I{}^{\mu}R_{\lambda}{}^{\nu}\right)e^K{}_{\tau}\partial_{\nu}e_K{}^{\tau}\\ &+e_I{}^{\tau}\Gamma^{\nu}{}_{\nu\tau}R_{\lambda}{}^{\mu}+e_I{}^{\nu}\Gamma^{\tau}{}_{\nu\lambda}R_{\tau}{}^{\mu}-e_I{}^{\mu}\Gamma^{\tau}{}_{\nu\lambda}R_{\tau}{}^{\nu}-e_I{}^{\tau}\Gamma^{\mu}{}_{\nu\tau}R_{\lambda}{}^{\nu}]\\ &+2\alpha l^{-2}[\left(e_I{}^{\nu}e_J{}^{\tau}-e_J{}^{\nu}e_I{}^{\tau}\right)\Gamma^{\mu}{}_{\nu\tau}R+\left(e_J{}^{\mu}e_I{}^{\nu}-e_I{}^{\mu}e_J{}^{\nu}\right)\left(\Gamma^{\lambda}{}_{\lambda\nu}R-\partial_{\nu}R\right)\\ &+\left(e_I{}^{\nu}e_J{}^{\mu}-e_I{}^{\mu}e_J{}^{\nu}\right)Re^K{}_{\tau}\partial_{\nu}e_K{}^{\tau}]+4\gamma l^{-2}e_{I\nu}e_J{}^{\tau}T^{\nu\mu}{}_{\tau}\\ &+\frac{1}{2}l^{-2}[\left(e_I{}^{\nu}e_J{}^{\tau}-e_J{}^{\nu}e_I{}^{\tau}\right)\Gamma^{\mu}{}_{\nu\tau}+\left(e_I{}^{\nu}e_J{}^{\mu}-e_I{}^{\mu}e_J{}^{\nu}\right)\left(\Gamma^{\lambda}{}_{\lambda\nu}+e^K{}_{\tau}\partial_{\nu}e_K{}^{\tau}\right)]\}e.\end{aligned} \tag{17}$$

That may be, the two main field equations are rather complicated. They really look nothing like the familiar, well-analyzed equations of GR. To help understand the significance of these new relations, and to use our previous experience, we will do a translation of (16,17) into a certain effective Riemannian form–transcribing from quantities expressed in terms of the tetrad $e_I{}^\mu$ and spin connection $\Gamma^{IJ}{}_\mu$ into the ones expressed in terms of the metric $g_{\mu\nu}$ and torsion $T^\lambda{}_{\mu\nu}$ (or contortion $K^\lambda{}_{\mu\nu}$).

As is well-known, the affine connection $\Gamma^\lambda{}_{\mu\nu}$ can be represented in the form

$$\begin{aligned}\Gamma^\lambda{}_{\mu\nu} &= e_I{}^\lambda \partial_\mu e^I{}_\nu + e_J{}^\lambda e^I{}_\nu \Gamma^J{}_{I\mu} \\ &= \left\{{}_\mu{}^\lambda{}_\nu\right\} + K^\lambda{}_{\mu\nu},\end{aligned} \tag{18}$$

where $\left\{{}_\mu{}^\lambda{}_\nu\right\}$, $K^\lambda{}_{\mu\nu}$ are the Christoffel symbol and the contortion, separately, with

$$\begin{aligned}K^\lambda{}_{\mu\nu} &= -\frac{1}{2}\left(T^\lambda{}_{\mu\nu} + T_{\mu\nu}{}^\lambda + T_{\nu\mu}{}^\lambda\right), \\ T^\lambda{}_{\mu\nu} &= e_I{}^\rho T^I{}_{\mu\nu} = \Gamma^\lambda{}_{\mu\nu} - \Gamma^\lambda{}_{\nu\mu}.\end{aligned} \tag{19}$$

Accordingly the curvature can be represented as

$$\begin{aligned}R^\rho{}_{\sigma\mu\nu} &= e_I{}^\rho e^J{}_\sigma R^I{}_{J\mu\nu} = \partial_\mu \Gamma^\rho{}_{\sigma\nu} - \partial_\nu \Gamma^\rho{}_{\sigma\mu} + \Gamma^\rho{}_{\lambda\mu}\Gamma^\lambda{}_{\sigma\nu} - \Gamma^\rho{}_{\lambda\nu}\Gamma^\lambda{}_{\sigma\mu}, \\ &= R^\rho_{\{\}\sigma\mu\nu} + \partial_\mu K^\rho{}_{\sigma\nu} - \partial_\nu K^\rho{}_{\sigma\mu} + K^\rho{}_{\lambda\mu}K^\lambda{}_{\sigma\nu} - K^\rho{}_{\lambda\nu}K^\lambda{}_{\sigma\mu} \\ &\quad + \left\{{}_\lambda{}^\rho{}_\mu\right\} K^\lambda{}_{\sigma\nu} - \left\{{}_\lambda{}^\rho{}_\nu\right\} K^\lambda{}_{\sigma\mu} + \left\{{}_\sigma{}^\lambda{}_\nu\right\} K^\rho{}_{\lambda\mu} - \left\{{}_\sigma{}^\lambda{}_\mu\right\} K^\rho{}_{\lambda\nu},\end{aligned} \tag{20}$$

where

$$R^\rho_{\{\}\sigma\mu\nu} = \partial_\mu \left\{{}_\sigma{}^\rho{}_\nu\right\} - \partial_\nu \left\{{}_\sigma{}^\rho{}_\mu\right\} + \left\{{}_\lambda{}^\rho{}_\mu\right\}\left\{{}_\sigma{}^\lambda{}_\nu\right\} - \left\{{}_\lambda{}^\rho{}_\nu\right\}\left\{{}_\sigma{}^\lambda{}_\mu\right\},$$

is the curvature of the Christoffel symbol.

III. COSMOLOGICAL EQUATIONS

For the space flat Friedmann-Robertson-Walker metric

$$g_{\mu\nu} = \mathrm{diag}\left(-1, a(t)^2, a(t)^2, a(t)^2\right), \tag{21}$$

we have

$$\begin{aligned}\left\{{}_0{}^0{}_0\right\} &= 0, \left\{{}_0{}^0{}_i\right\} = \left\{{}_i{}^0{}_0\right\} = 0, \left\{{}_i{}^0{}_j\right\} = a\,\dot{a}\,\delta_{ij}, \\ \left\{{}_0{}^i{}_0\right\} &= 0, \left\{{}_j{}^i{}_0\right\} = \left\{{}_0{}^i{}_j\right\} = \frac{\dot{a}}{a}\delta^i_j, \left\{{}_j{}^i{}_k\right\} = 0, i, j, k, ... = 1, 2, 3.\end{aligned} \tag{22}$$

The non-vanishing torsion components with holonomic indices are given by two functions h and f [13]:

$$T_{110} = T_{220} = T_{330} = a^2 h, \quad T_{123} = T_{231} = T_{312} = a^3 f, \tag{23}$$

and then contortion components are

$$\begin{aligned} K^1{}_{10} &= K^2{}_{20} = K^3{}_{30} = 0, \\ K^1{}_{01} &= K^2{}_{02} = K^3{}_{03} = h, \\ K^0{}_{11} &= K^0{}_{22} = K^0{}_{22} = a^2 h, \\ K^1{}_{23} &= K^2{}_{31} = K^3{}_{12} = -\frac{1}{2} a f, \\ K^1{}_{32} &= K^2{}_{13} = K^3{}_{21} = \frac{1}{2} a f. \end{aligned} \tag{24}$$

Among the torsion components, only the pseudotrace axial ingredient given by f couples to spinors in a minimal way. The scalar mode h of torsion could be considered as a "phantom" field, at least in the matter-dominated epoch, since it will not interact directly with matter; it only interacts indirectly via gravitation.

The non-vanishing components of the curvature $R^{\rho}_{\{\}\sigma\mu\nu}$ are

$$\begin{aligned} R^0_{\{\}101} &= R^0_{\{\}202} = R^0_{\{\}303} = a^2 \left(\dot{H} + H^2 + Hh + \dot{h}\right), \\ R^0_{\{\}123} &= -R^0_{\{\}213} = R^0_{\{\}312} = a^3 f (H + h), \\ R^1_{\{\}203} &= -R^1_{\{\}302} = R^2_{\{\}301} = -\frac{1}{2} a \left(H f + \dot{f}\right), \\ R^1_{\{\}212} &= R^1_{\{\}313} = R^2_{\{\}323} = a^2 \left(H^2 + 2Hh + h^2 - \frac{1}{4} f^2\right), \end{aligned} \tag{25}$$

$$\begin{aligned} R_{\{\}00} &= -3\dot{H} - 3\dot{h} - 3H^2 - 3Hh, \\ R_{\{\}11} &= a^2 \left(\dot{H} + 3H^2 + 5Hh + \dot{h} + 2h^2 - \frac{1}{2} f^2\right), \end{aligned} \tag{26}$$

$$R_{\{\}} = 6\dot{H} + 12H^2 + 18Hh + 6\dot{h} + 6h^2 - \frac{3}{2} f^2, \tag{27}$$

where $H = \dot{a}(t)/a(t)$ is the Hubble parameter. Using these results and (16—20) we can compute

$$\begin{aligned} e_{I0} \frac{\delta \mathcal{L}}{\delta e_I{}^0} = {} & l^{-2} \{(\beta + 3\alpha) [-12 \left(\dot{H} + \dot{h}\right)^2 - 24 \left(\dot{H} + \dot{h}\right) H (H + h) \\ & + 12h (h + 2H) (h + H)^2 - 6 (h + H)^2 f^2 + \frac{3}{4} f^4] \\ & + \gamma \left(18h^2 + 6f^2\right) + 3H^2 + 6Hh + 3h^2 - \frac{3}{4} f^2 - 12 l^{-2}\} e, \end{aligned} \tag{28}$$

$$
\begin{aligned}
e_{I1}\frac{\delta\mathcal{L}}{\delta e_I{}^1} \;=\; & -l^{-2}a^2\{(\beta+3\alpha)\,[-4\left(\dot{H}+\dot{h}\right)^2-8\left(\dot{H}+\dot{h}\right)\left(H^2+Hh\right)\\
& +4h\,(h+2H)\,(h+H)^2-2\,(h+H)^2\,f^2+\frac{1}{4}f^4]\\
& -2\gamma\left(2\,\dot{h}+8Hh+h^2+f^2\right)+2\left(\dot{H}+\dot{h}\right)+3H^2\\
& +4Hh+h^2-\frac{1}{4}f^2-12l^{-2}\}e,
\end{aligned}
\tag{29}
$$

$$
\begin{aligned}
\frac{\delta\mathcal{L}}{\delta\Gamma^{\,01}{}_1} \;=\; & -2a^{-1}l^{-2}\{(\beta+6\alpha)\left(\ddot{H}+\ddot{h}\right)+3\,(\beta+4\alpha)\left(hH^2+2H\,\dot{H}+2h\,\dot{H}\right)\\
& +(5\beta+18\alpha)\left(H\,\dot{h}+h\,\dot{h}+h^2H\right)+(\beta+3\alpha)\left(2h^3-f\,\dot{f}-\frac{1}{2}hf^2\right)+\frac{1}{4}h\}e,
\end{aligned}
\tag{30}
$$

$$
\begin{aligned}
& \frac{\delta\mathcal{L}}{\delta\Gamma^{12}{}_3}=a^{-1}l^{-2}f\{2\,(\beta+6\alpha)\left(\dot{H}+\dot{h}\right)+6\,(\beta+4\alpha)\,H^2\\
& +2\,(5\beta+18\alpha)\,Hh+(\beta\;+3\alpha)\left(4h^2-f^2\right)\\
& -4\gamma+\frac{1}{2}\}e,
\end{aligned}
\tag{31}
$$

Suppose the matter source is a fluid characterized by the density ρ the pressure p and the spin $s_{IJ}{}^{\mu}$. The system of field equations (15) consists of four independent ones:

$$
\begin{aligned}
e_{I0}\frac{\delta\mathcal{L}}{\delta e_I{}^0} &= -e_{I0}\frac{\delta\mathcal{L}_\psi}{\delta e_I{}^0}=\rho,\\
e_{I1}\frac{\delta\mathcal{L}}{\delta e_I{}^1} &= -e_{I1}\frac{\delta\mathcal{L}_\psi}{\delta e_I{}^1}=g_{11}p,\\
\frac{\delta\mathcal{L}}{\delta\Gamma^{01}{}_1} &= -\frac{\delta\mathcal{L}_\psi}{\delta\Gamma^{01}{}_1}=e_1{}^1s_{01}{}^1,\\
\frac{\delta\mathcal{L}}{\delta\Gamma^{12}{}_3} &= -\frac{\delta\mathcal{L}_\psi}{\delta\Gamma^{12}{}_3}=e_3{}^3s_{12}{}^3.
\end{aligned}
\tag{32}
$$

Using (28-31) the Lagrange equations (32) can be written as

$$
\begin{aligned}
& (\beta+3\alpha)\,[-12\left(\dot{H}+\dot{h}\right)^2-24\left(\dot{H}+\dot{h}\right)H\,(H+h)\\
& +12h\,(h+2H)\,(h+H)^2-6\,(h+H)^2\,f^2+\frac{3}{4}f^4]\\
& +\gamma\left(18h^2+6f^2\right)+3H^2+6Hh+3h^2-\frac{3}{4}f^2-12l^{-2}-l^2\rho=0,
\end{aligned}
\tag{33}
$$

$$
\begin{aligned}
& (\beta+3\alpha)\,[-4\left(\dot{H}+\dot{h}\right)^2-8\left(\dot{H}+\dot{h}\right)\left(H^2+Hh\right)\\
& +4h\,(h+2H)\,(h+H)^2-2\,(h+H)^2\,f^2+\frac{1}{4}f^4]\\
& +2\gamma\left(2\,\dot{h}+8Hh+h^2+f^2\right)-2\left(\dot{H}+\dot{h}\right)-3H^2\\
& -4Hh-h^2+\frac{1}{4}f^2+12l^{-2}+l^2p=0,
\end{aligned}
\tag{34}
$$

$$-2\{(\beta+6\alpha)\left(\ddot{H}+\ddot{h}\right)+3(\beta+4\alpha)\left(hH^2+2H\dot{H}+2h\dot{H}\right)+(5\beta+18\alpha)\left(H\dot{h}+h\dot{h}+h^2H\right)$$
$$+(\beta+3\alpha)\left(2h^3-f\dot{f}-\frac{1}{2}hf^2\right)+\frac{1}{4}h\}-l^2 s_{01}{}^1=0, \tag{35}$$

$$f\{2(\beta+6\alpha)\left(\dot{H}+\dot{h}\right)+6(\beta+4\alpha)H^2$$
$$+2(5\beta+18\alpha)Hh+(\beta+3\alpha)\left(4h^2-f^2\right)-4\gamma+\frac{1}{2}\}-l^2 s_{12}{}^3=0. \tag{36}$$

Assuming $s_{\mu\nu}{}^{\lambda}=0$ (i.e., the source spin current is negligible), the Eq. (36) reads

$$f\{2(\beta+6\alpha)\left(\dot{H}+\dot{h}\right)+6(\beta+4\alpha)H^2$$
$$+2(5\beta+18\alpha)Hh+(\beta+3\alpha)\left(4h^2-f^2\right)-4\gamma+\frac{1}{2}\}=0, \tag{37}$$

and gives

$$f=0, \tag{38}$$

or

$$2(\beta+6\alpha)\left(\dot{H}+\dot{h}\right)+6(\beta+4\alpha)H^2$$
$$+2(5\beta+18\alpha)Hh+(\beta+3\alpha)\left(4h^2-f^2\right)-4\gamma+\frac{1}{2}=0. \tag{39}$$

Therefore, we have two cases.

In the first case, $f=0$, the Eqs (33) and (34) read

$$(\beta+3\alpha)[-\left(\dot{H}+\dot{h}\right)^2-2\left(\dot{H}+\dot{h}\right)H(H+h)+h(h+2H)(h+H)^2]$$
$$+\frac{3}{2}\gamma h^2+\frac{1}{4}H^2+\frac{1}{2}Hh+\frac{1}{4}h^2-l^{-2}-\frac{l^2\rho}{12}=0, \tag{40}$$

and

$$(\beta+3\alpha)[-\left(\dot{H}+\dot{h}\right)^2-2\left(\dot{H}+\dot{h}\right)\left(H^2+Hh\right)+h(h+2H)(h+H)^2]$$
$$+\frac{1}{2}\gamma\left(2\dot{h}+8Hh+h^2\right)-\frac{1}{2}\left(\dot{H}+\dot{h}\right)-\frac{3}{4}H^2-Hh-\frac{h^2}{4}+3l^{-2}+\frac{l^2 p}{4}=0, \tag{41}$$

which lead to

$$\dot{H}=(2\gamma-1)\dot{h}-2H^2+(8\gamma-3)Hh-(2\gamma+1)h^2+8l^{-2}+\frac{l^2}{6}(\rho+3p), \tag{42}$$

and

$$
\begin{aligned}
&-4\gamma^2 \dot{h}^2 + \gamma\left(4H^2 + 8(1-4\gamma)Hh + 4(2\gamma+1)h^2 - \frac{2}{3}l^2(\rho - 3p) - \frac{32}{l^2}\right)\dot{h} \\
&+16H^3h\gamma + \left(28\gamma - 64\gamma^2\right)H^2h^2 + 8(4\gamma+1)\gamma h^3 H - 4(\gamma+1)\gamma h^4 \\
&+\left(\frac{16}{l^2} + \frac{1}{3}l^2(\rho - 3p)\right)H^2 + \frac{1}{4(\beta+3\alpha)}H^2 + (1-4\gamma)\left(\frac{32}{l^2} + \frac{2}{3}l^2(\rho-3p)\right)Hh \\
&+\frac{1}{2(\beta+3\alpha)}Hh + (1+2\gamma)\left(\frac{16}{l^2} + \frac{1}{3}l^2(\rho-3p)\right)h^2 + \frac{(6\gamma+1)}{4(\beta+3\alpha)}h^2 \\
&-\frac{l^2\rho}{12(\beta+3\alpha)} - \frac{8}{3}(\rho-3p) - \frac{1}{36}l^4(\rho-3p)^2 - \frac{1}{(\beta+3\alpha)l^2} - \frac{64}{l^4} \\
= \;& 0. \qquad (43)
\end{aligned}
$$

So we have the equations (42), (43) and

$$
\begin{aligned}
&(\beta+6\alpha)\left(\ddot{H} + \ddot{h}\right) + 3(\beta+4\alpha)\left(hH^2 + 2H\dot{H} + 2h\dot{H}\right) + (5\beta+18\alpha)\left(H\dot{h} + h\dot{h} + h^2H\right) \\
&+2(\beta+3\alpha)h^3 + \frac{1}{4}h = 0, \qquad (44)
\end{aligned}
$$

for the unknown functions H and h.

In the second case, f satisfies the condition (39). The Eqs. (33) and (34) yield

$$
\dot{H} = (2\gamma-1)\dot{h} - 2H^2 + (8\gamma-3)Hh - (2\gamma+1)h^2 + \frac{1}{4}f^2 + 8l^{-2} + \frac{l^2}{6}(\rho+3p), \qquad (45)
$$

and

$$
\begin{aligned}
&-4\gamma^2\dot{h}^2 + \gamma\left(4H^2 + 8(1-4\gamma)Hh + 4(1+2\gamma)h^2 - f^2 - \frac{2}{3}l^2B - \frac{32}{l^2}\right)\dot{h} \\
&+16H^3h\gamma + 4\gamma(7-16\gamma)H^2h^2 + 8\gamma(1+4\gamma)h^3H - 4\gamma(1+\gamma)h^4 - 4\gamma Hhf^2 + \gamma h^2 f^2 \\
&+\left(\frac{16}{l^2} + \frac{1}{3}l^2(\rho+3p)\right)H^2 + \frac{1}{4(\beta+3\alpha)}H^2 + (1-4\gamma)\left(\frac{32}{l^2} + \frac{2}{3}l^2B\right)Hh + \frac{1}{2(\beta+3\alpha)}Hh \\
&+\frac{16}{l^2}(1+2\gamma)h^2 + \frac{1}{3}(1+2\gamma)l^2(\rho+3p)h^2 + \frac{6\gamma+1}{4(\beta+3\alpha)}h^2 - \left(\frac{4}{l^2} + \frac{1}{12}l^2(\rho+3p)\right)f^2 \\
&+\frac{8\gamma-1}{16(\beta+3\alpha)}f^2 - \frac{1}{(\beta+3\alpha)}l^{-2} - \frac{l^2}{12(\beta+3\alpha)}\rho - \frac{8}{3}(\rho+3p) - \frac{1}{36}l^4(\rho+3p)^2 - \frac{64}{l^4} \\
= \;& 0. \qquad (46)
\end{aligned}
$$

The Eqs. (45) and (39) gives

$$
\begin{aligned}
f^2 \;=\; & 8\gamma\frac{(\beta+6\alpha)}{\beta}\dot{h} + 4H^2 + 8\left(1 + 4\gamma\frac{(\beta+6\alpha)}{\beta}\right)Hh + \left(4 - 8\gamma\frac{(\beta+6\alpha)}{\beta}\right)h^2 \\
& + \frac{1-8\gamma}{\beta} + \frac{32(\beta+6\alpha)}{\beta l^2} + \frac{2(\beta+6\alpha)}{3\beta}l^2(\rho+3p). \qquad (47)
\end{aligned}
$$

Substituting into (45) and (46) yields

$$
\begin{aligned}
& -12\gamma^2\frac{\beta+4\alpha}{\beta}\dot{h}^2 \\
& +\left(-32\frac{\gamma}{\beta}\left(\gamma\left(\beta+6\alpha\right)-2\left(\beta+3\alpha\right)\right)Hh+8\frac{\gamma}{\beta}\left(\gamma\left(\beta+6\alpha\right)+2\left(\beta+3\alpha\right)\right)h^2\right)\dot{h} \\
& +\left(\frac{8\gamma-1}{\beta}\left(\frac{\gamma\left(\beta+6\alpha\right)}{2\left(\beta+3\alpha\right)}+1\right)-\frac{4}{l^2}\frac{17\beta+48\alpha}{\beta}-\frac{17\beta+48\alpha}{12\beta}l^2\left(\rho+3p\right)\right)\dot{h} \\
& -192\gamma^2\frac{\beta+4\alpha}{\beta}H^2h^2+96\gamma^2\frac{\beta+4\alpha}{\beta}Hh^3-12\gamma^2\frac{\beta+4\alpha}{\beta}h^4 \\
& +\frac{2\gamma}{\beta+3\alpha}H^2+\left[2\gamma\frac{24\gamma\beta+96\gamma\alpha-\beta-12\alpha}{\beta\left(\beta+3\alpha\right)}-384\gamma\frac{\beta+4\alpha}{\beta l^2}-8\gamma\frac{\beta+4\alpha}{\beta}l^2\left(\rho+3p\right)\right]Hh \\
& +\left[-\gamma\frac{-5\beta+12\gamma\beta+48\gamma\alpha-6\alpha}{\beta\left(\beta+3\alpha\right)}+96\gamma\frac{\beta+4\alpha}{\beta l^2}+2\gamma\frac{\beta+4\alpha}{\beta}l^2\left(\rho+3p\right)\right]h^2 \\
& +\frac{1-8\gamma}{\beta}\frac{8\gamma-1}{16\left(\beta+3\alpha\right)}+\frac{48\gamma\beta+192\gamma\alpha-7\beta-24\alpha}{\beta\left(\beta+3\alpha\right)l^2}-192\frac{\beta+4\alpha}{\beta l^4} \\
& -\frac{l^2}{12\left(\beta+3\alpha\right)}\rho+\left(\frac{8\gamma-1}{8}\frac{\beta+4\alpha}{\beta\left(\beta+3\alpha\right)}l^2-8\frac{\beta+4\alpha}{\beta}\right)\left(\rho+3p\right)-\frac{\beta+4\alpha}{12\beta}l^4\left(\rho+3p\right)^2 \\
= \; & 0. \qquad (48)
\end{aligned}
$$

and

$$
\begin{aligned}
\dot{H} \; = \; & \left(\frac{4\gamma\left(\beta+3\alpha\right)}{\beta}-1\right)\dot{h}-H^2+\left(16\gamma\frac{\beta+3\alpha}{\beta}-1\right)Hh-4\gamma\frac{\beta+3\alpha}{\beta}h^2 \\
& +\frac{1-8\gamma}{4\beta}+16\frac{\beta+3\alpha}{\beta l^2}+\frac{\beta+3\alpha}{3\beta}l^2\left(\rho-3p\right). \qquad (49)
\end{aligned}
$$

Differentiating (47) gives

$$
\begin{aligned}
-f\dot{f} \; = \; & -4\gamma\frac{\left(\beta+6\alpha\right)}{\beta}\ddot{h}-4\left(1+4\gamma\frac{\left(\beta+6\alpha\right)}{\beta}\right)h\dot{H}-4H\dot{H}-4\left(1+4\gamma\frac{\left(\beta+6\alpha\right)}{\beta}\right)H\dot{h}-\left(4-8\gamma\frac{\left(\beta+6\alpha\right)}{\beta}\right)h\dot{h} \\
& -\frac{\left(\beta+6\alpha\right)}{3\beta}l^2\left(\dot{\rho}+3\dot{p}\right).
\end{aligned}
$$

Substituting it and (47) into (35) and letting ${s_{01}}^1=0$ give

$$
\begin{aligned}
& \ddot{H}+\left(1-4\gamma\frac{\left(\beta+3\alpha\right)}{\beta}\right)\ddot{h}+2H\dot{H}+2\left(1-\frac{8\gamma}{\beta}\left(\beta+3\alpha\right)\right)h\dot{H} \\
& +\left(1-16\gamma\frac{\left(\beta+3\alpha\right)}{\beta}\right)H\dot{h}+\left(4\gamma\frac{\left(\beta+3\alpha\right)}{\beta}+1\right)h\dot{h} \\
& +hH^2+\left(1-16\gamma\frac{\left(\beta+3\alpha\right)}{\beta}\right)Hh^2+4\gamma\frac{\left(\beta+3\alpha\right)}{\beta}h^3 \\
& +\left(4\gamma\frac{\beta+3\alpha}{\beta\left(\beta+6\alpha\right)}-\frac{1}{4\beta}\right)h-\frac{16\left(\beta+3\alpha\right)}{\beta l^2}h-\frac{\left(\beta+3\alpha\right)}{3\beta}l^2h\left(\rho+3p\right) \\
& -\frac{\left(\beta+3\alpha\right)}{3\beta}l^2\left(\dot{\rho}+3\dot{p}\right) \\
= \; & 0. \qquad (50)
\end{aligned}
$$

So we have the equations (48), (49), and (50) for the unknown functions H and h. The unknown function f is given by (47).

IV. TWO SPECIFIC MODELS

In order to emphasize the geometrical nature of the effect of acceleration of cosmological expansion we concentrate on vacuum solutions in two specific cases and discuss only the acceleration solutions.

A. When $\beta = -3\alpha$

This corresponds to conformal (Weyl) gravity which has been investigated by numerous authors [recent, see 2 and 9] but it must be pointed out that the principle and structure between the theory here and higher-derivative gravity in Mannheim's theory are quite different.

According to last section, the equation (37) gives two cases.

In the first case $f = 0$, the functions H and h now satisfy the equations (40), (41) and (44), i.e.,

$$(6\gamma + 1)\,h^2 + H^2 + 2Hh - 4l^{-2} = 0, \tag{51}$$

$$(4\gamma - 2)\,\dot{h} - 2\,\dot{H} + (16\gamma - 4)\,Hh + (2\gamma - 1)\,h^2 - 3H^2 + 12l^{-2} = 0, \tag{52}$$

$$\begin{aligned} &\ddot{H} + \ddot{h} + 2\,(H + h)\,\dot{H} \\ &+ (H + h)\,\dot{h} + hH^2 + h^2H + \frac{1}{12\alpha}h = 0. \end{aligned} \tag{53}$$

Eq. (51) has the roots

$$h = \frac{-H \pm \sqrt{-6\gamma H^2 + 4\,(6\gamma + 1)\,l^{-2}}}{(6\gamma + 1)}. \tag{54}$$

Eq. (52) gives

$$\dot{h} = \frac{1}{(2\gamma - 1)}\,\dot{H} - \frac{(8\gamma - 2)}{(2\gamma - 1)}Hh - \frac{1}{2}h^2 + \frac{3}{2\,(2\gamma - 1)}H^2 - \frac{6l^{-2}}{(2\gamma - 1)}, \tag{55}$$

and then

$$\ddot{h} = \frac{1}{(2\gamma - 1)}\,\ddot{H} - \frac{(8\gamma - 2)}{(2\gamma - 1)}h\,\dot{H} - \left(\frac{(8\gamma - 2)}{(2\gamma - 1)}H + \frac{1}{2}h\right)\dot{h} + \frac{3}{2\,(2\gamma - 1)}H\,\dot{H}\,. \tag{56}$$

Substituting (54), (55) and (56) into (53) yields

$$\begin{aligned}\ddot{H} &= -\left(\frac{48\gamma^3-50\gamma^2-7\gamma+2}{2\gamma(2\gamma-1)(6\gamma+1)}H \mp \frac{8\gamma-1}{4\gamma}\frac{\sqrt{-6\gamma H^2+4(6\gamma+1)l^{-2}}}{(6\gamma+1)}\right)\dot{H} + \frac{2\left(504\gamma^3+324\gamma^2-26\gamma-3\right)}{(6\gamma+1)^3(2\gamma-1)}H^3 \\ &+\frac{840\gamma^3-4\gamma^2-6\gamma-5}{(2\gamma-1)(6\gamma+1)^2\gamma l^2}H + \frac{2\gamma-1}{24\alpha\gamma(6\gamma+1)}H \\ &\mp\left(\frac{-476\gamma^2+2592\gamma^4+744\gamma^3-18\gamma+1}{4\gamma(6\gamma+1)^3(2\gamma-1)}H^2 + \frac{3(2\gamma+1)}{\gamma(6\gamma+1)^2 l^2} + \frac{2\gamma-1}{24\alpha\gamma(6\gamma+1)}\right)\sqrt{-6\gamma H^2+4(6\gamma+1)l^{-2}} \qquad (57)\end{aligned}$$

Let

$$\dot{H} = X.$$

We have the dynamical system

$$\begin{aligned}\dot{H} &= X, \\ \dot{X} &= -\left(\frac{48\gamma^3-50\gamma^2-7\gamma+2}{2\gamma(2\gamma-1)(6\gamma+1)}H \mp \frac{8\gamma-1}{4\gamma}\frac{\sqrt{-6\gamma H^2+4(6\gamma+1)l^{-2}}}{(6\gamma+1)}\right)X \\ &+AH^3+BH\mp(CH^2+D)\sqrt{-6\gamma H^2+4(6\gamma+1)l^{-2}}\end{aligned} \qquad (58)$$

where

$$\begin{aligned}A &= \frac{2\left(504\gamma^3+324\gamma^2-26\gamma-3\right)}{(6\gamma+1)^3(2\gamma-1)}, \\ B &= \frac{840\gamma^3-4\gamma^2-6\gamma-5}{(2\gamma-1)(6\gamma+1)^2\gamma l^2} + \frac{2\gamma-1}{24\alpha\gamma(6\gamma+1)}, \\ C &= \frac{-476\gamma^2+2592\gamma^4+744\gamma^3-18\gamma+1}{4\gamma(6\gamma+1)^3(2\gamma-1)}, \\ D &= \frac{3(2\gamma+1)}{\gamma(6\gamma+1)^2 l^2} + \frac{2\gamma-1}{24\alpha\gamma(6\gamma+1)}.\end{aligned} \qquad (59)$$

The Jacobian elements are

$$\frac{\partial \dot{H}}{\partial H} = 0, \frac{\partial \dot{H}}{\partial X} = 1,$$

$$\begin{aligned}\frac{\partial \dot{X}}{\partial H} &= \left(-\frac{48\gamma^3-50\gamma^2-7\gamma+2}{2\gamma(2\gamma-1)(6\gamma+1)} \mp \frac{3(8\gamma-1)H}{2(6\gamma+1)\sqrt{-6\gamma H^2+4(6\gamma+1)l^{-2}}}\right)X \\ &+AH^3+BH\mp(CH^2+D)\sqrt{-6\gamma H^2+4(6\gamma+1)l^{-2}},\end{aligned}$$

$$\frac{\partial \dot{X}}{\partial X} = -\frac{48\gamma^3-50\gamma^2-7\gamma+2}{2\gamma(2\gamma-1)(6\gamma+1)}H \pm \frac{8\gamma-1}{4\gamma(6\gamma+1)}\sqrt{-6\gamma H^2+4(6\gamma+1)l^{-2}}. \qquad (60)$$

The critical point equations are

$$\begin{aligned} X &= 0, \\ AH^3 + BH \mp \left(CH^2 + D\right)\sqrt{-6\gamma H^2 + 4\left(6\gamma + 1\right)l^{-2}} &= 0. \end{aligned} \tag{61}$$

Rationalization gives

$$H^6 + aH^4 + bH^2 + c = 0, \tag{62}$$

where

$$\begin{aligned} a &= \frac{2\left(l^2AB - 12C^2\gamma - 2C^2 + 6CD\gamma l^2\right)}{\left(A^2 + 6C^2\gamma\right)l^2}, \\ b &= \frac{B^2l^2 - 48\gamma DC - 8CD + 6D^2\gamma l^2}{\left(A^2 + 6C^2\gamma\right)l^2}, \\ c &= -\frac{4\left(6\gamma + 1\right)}{\left(A^2 + 6C^2\gamma\right)l^2}D^2. \end{aligned} \tag{63}$$

The equation (62) has the roots

$$\begin{aligned} H_1^2 &= \left(-\frac{q}{2} + \sqrt{\Delta}\right)^{1/3} + \left(-\frac{q}{2} - \sqrt{\Delta}\right)^{1/3} - \frac{a}{3}, \\ H_2^2 &= \left(-\frac{q}{2} + \sqrt{\Delta}\right)^{1/3}\omega + \left(-\frac{q}{2} - \sqrt{\Delta}\right)^{1/3}\omega^2 - \frac{a}{3}, \\ H_3^2 &= \left(-\frac{q}{2} + \sqrt{\Delta}\right)^{1/3}\omega^2 + \left(-\frac{q}{2} - \sqrt{\Delta}\right)^{1/3}\omega - \frac{a}{3}. \end{aligned} \tag{64}$$

where

$$\begin{aligned} p &= \left(-\frac{1}{3}a^2 + b\right), \\ q &= \frac{2}{27}a^3 - \frac{1}{3}ba + c, \end{aligned}$$

$$\Delta = \left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3, \tag{65}$$

and

$$\omega = \frac{-1 + \sqrt{3}i}{2}. \tag{66}$$

Now we have the critical points

$$
\begin{aligned}
H_1 &= \pm\sqrt{\left(-\frac{q}{2}+\sqrt{\Delta}\right)^{1/3}+\left(-\frac{q}{2}-\sqrt{\Delta}\right)^{1/3}-\frac{a}{3}}, X_1 = 0,\\
H_2 &= \pm\sqrt{\left(-\frac{q}{2}+\sqrt{\Delta}\right)^{1/3}\omega+\left(-\frac{q}{2}-\sqrt{\Delta}\right)^{1/3}\omega^2-\frac{a}{3}}, X_2 = 0,\\
H_3 &= \pm\sqrt{\left(-\frac{q}{2}+\sqrt{\Delta}\right)^{1/3}\omega^2+\left(-\frac{q}{2}-\sqrt{\Delta}\right)^{1/3}\omega-\frac{a}{3}}, X_3 = 0.
\end{aligned}
$$

In order to analyze their stability we give the parameter α and γ specific values and then obtain the results:

When

$$\alpha = \frac{1}{32}l^2, \gamma = -\frac{1}{4},$$

the equations (62) become

$$431H^6l^6 - 13700H^4l^4 + 15798H^2l^2 + 3600 = 0.$$

It has the roots

$$H^2 \approx 1.4024/l^2, H^2 \approx -.19478/l^2 - 2.0\times10^{-9}i/l^2, H^2 \approx 10.596/l^2 + .92212i/l^2,$$

the first root $H^2 = 1.4024/l^2$ corresponds a positive critical point

$$H = 1.1842/l, X = 0.$$

At this point, for

$$h = \frac{-H+\sqrt{-6\gamma H^2+4\left(6\gamma+1\right)l^{-2}}}{(6\gamma+1)} = \frac{1.725}{l},$$

the dynamic system (58) reads

$$
\begin{aligned}
\dot{H} &= X,\\
\dot{X} &= -\left(\frac{1}{3}H+3\sqrt{\left(6H^2-\frac{8}{l^2}\right)}\right)X\\
&\quad +\frac{508}{3}H^3-\frac{196}{l^2}H-\left(\frac{412}{3}H^2-\frac{40}{l^2}\right)\sqrt{\frac{3}{2}H^2-\frac{2}{l^2}}.
\end{aligned}
$$

The Jacobian

$$\mathcal{M} = \begin{pmatrix} 0 & 1 \\ -\frac{430.76}{l^2} & -\frac{2.325}{l} \end{pmatrix}$$

has the eigenvalues $-1.1625/l - 20.7222i/l$, $-1.1625/l + 20.7222i/l$. Therefore, the critical point

$$H = 1.1842/l, X = 0,$$

is stable, where

$$h = \frac{1.725}{l}, f = 0.$$

When

$$\alpha = \left(\frac{1}{32}l^2\right), \gamma = \left(\frac{1}{4}\right),$$

the equations (62) become

$$1827H^6l^6 + 5226H^4l^4 + 2579H^2l^2 - 2312 = 0.$$

It has roots

$$H^2 \approx 0.4412/l^2, H^2 \approx -1.6508/l^2 + 0.37826i/l^2, H^2 \approx -1.6508/l^2 - 0.37826i/l^2,$$

the first root $H^2 = 0.4412/l^2$ corresponds a positive critical point

$$H = 0.66423/l, X = 0.$$

At this point, for

$$h = \frac{-H - \sqrt{-6\gamma H^2 + 4(6\gamma + 1)l^{-2}}}{(6\gamma + 1)} = -\frac{1.488}{l},$$

the dynamic system (58) reads

$$\begin{aligned} \dot{H} &= X, \\ \dot{X} &= -\left(\frac{17}{5}H + \frac{1}{5}\sqrt{\left(-6H^2 + \frac{40}{l^2}\right)}\right)X \\ &\quad -\frac{596}{125}H^3 - \frac{692}{75l^2}BH + \left(\frac{184}{125}H^2 + \frac{136}{75l^2}\right)\sqrt{-\frac{3}{2}H^2 + \frac{10}{l^2}}. \end{aligned}$$

The Jacobian

$$\mathcal{M} = \begin{pmatrix} 0 & 1 \\ -\frac{10.365}{l^2} & -\frac{3.4807}{l} \end{pmatrix}$$

has the eigenvalues $-1.74035/l - 2.70854i/l$, $-1.74035/l + 2.70854i/l$. Therefore, the critical point

$$H = 0.66423/l, X = 0,$$

is stable, there

$$h = -\frac{1.488}{l},\ f = 0.$$

For the two points

$$X = \dot{H} = 0,$$

which corresponds to a de Sitter spacetime.

Following Lu and Pope [1], we chose $\alpha = -\frac{1}{2\Lambda}$, which means

$$\alpha = -\frac{l^2}{48}.$$

In contrast with them we deal with a de Sitter spacetime with torsion and the gravitational Lagrangian including a term $\gamma l^{-2} T^{\mu}{}_{\nu\rho} T_{\mu}{}^{\nu\rho}$. When we chose

$$\gamma = -\frac{1}{4},$$

(59) and (63) give, respectively,

$$A = \frac{508}{3}, B = -\frac{156}{l^2}, C = \frac{412}{3}, D = 0,$$

and

$$a = -\frac{17000}{431l^2}, b = \frac{27378}{431l^4}, c = 0.$$

The dynamical system (58) becomes

$$\begin{aligned}
\dot{H} &= X, \\
\dot{X} &= -\left(\frac{1}{3}H \pm \sqrt{\frac{3}{2}H^2 - \frac{2}{l^2}}\right)X + \frac{508}{3}H^3 - \frac{156}{l^2}H \mp \frac{412}{3}H^2\sqrt{\frac{3}{2}H^2 - \frac{2}{l^2}}
\end{aligned}$$

(62) becomes

$$\left(H^4 - \frac{17000}{431l^2}H^2 + \frac{27378}{431l^4}\right)H^2 = 0,$$

and has the roots

$$H_1 = 0, H_2 = \sqrt{\frac{8500 - 103\sqrt{5698}}{431}}/l, H_3 = \sqrt{\frac{8500 + 103\sqrt{5698}}{431}}/l.$$

Therefore we get three critical points

$$
\begin{aligned}
H_1 &= 0, X_1 = 0,\\
H_2 &= \sqrt{\frac{8500 - 103\sqrt{5698}}{431}}/l, X_2 = 0,\\
H_3 &= \sqrt{\frac{8500 + 103\sqrt{5698}}{431}}/l, X_3 = 0.
\end{aligned}
$$

At the point

$$H_1 = 0, X_1 = 0,$$

the Jacobian

$$M = \begin{pmatrix} 0 & 1 \\ -\frac{156}{l^2} & \mp\frac{6\sqrt{2}i}{l} \end{pmatrix}$$

has the eigenvalues

$$\left(-3\sqrt{2} + \sqrt{174}\right) i/l, \left(-3\sqrt{2} - \sqrt{174}\right) i/l.$$

This point is a center.

At the point

$$H_2 = \sqrt{\frac{8500 - 103\sqrt{5698}}{431}}/l, X_2 = 0,$$

the Jacobian

$$M = \begin{pmatrix} 0 & 1 \\ -\frac{180.44}{l^2} & -\frac{4.7728}{l} \end{pmatrix}$$

has the eigenvalues

$$-2.3864/l + 13.2191i/l, -2.3864/l - 13.2191i/l.$$

This is a stable critical point, where

$$h = 1.1472/l, f = 0.$$

At the point

$$H_3 = \sqrt{\frac{8500 + 103\sqrt{5698}}{431}}/l, X_3 = 0$$

the Jacobian

$$M=\begin{pmatrix}0 & 1\\ \frac{84.5}{l^2} & -\frac{46.4}{l}\end{pmatrix}$$

has the eigenvalues

$$-48.15/l, 1.755/l.$$

This is a unstable critical point.

If we chose

$$\gamma=\frac{1}{4},$$

(59) and (63) give, respectively,

$$A=-\frac{596}{125}, B=-\frac{164}{25l^2}, C=\frac{184}{125}, D=\frac{112}{25l^2},$$

and

$$a=\frac{474}{203l^2}, b=-\frac{459}{203l^4}, c=-\frac{224}{29l^6}.$$

The dynamical system (58) becomes

$$\begin{aligned}\dot{H} &= X,\\ \dot{X} &= -\left(\frac{17}{5}H\mp\frac{2}{5}\sqrt{-\frac{3}{2}H^2+\frac{10}{l^2}}\right)X-\frac{596}{125}H^3-\frac{164}{25l^2}H\mp(\frac{184}{125}H^2+\frac{112}{25l^2})\sqrt{-\frac{3}{2}H^2+\frac{10}{l^2}}.\end{aligned}$$

(62) becomes

$$H^6+\frac{474}{203l^2}H^4-\frac{459}{203l^4}H^2-\frac{224}{29l^6}=0$$

It has a real root

$$H=1.30134/l.$$

At the critical point

$$H=1.30134/l, X=0,$$

the Jacobian

$$M=\begin{pmatrix}0 & 1\\ -\frac{36.265}{l^2} & -\frac{3.3321}{l}\end{pmatrix}$$

has the eigenvalues

$$-1.666/l - 5.787i/l, -1.666/l + 5.787i/l.$$

This is a stable critical point, where

$$h = -1.613/l,\ f = 0.$$

In the second case, the functions H, h and f satisfy the equations (33-36) which now read

$$f^2 = \frac{4}{1-8\gamma}H^2 + \frac{8}{1-8\gamma}Hh + \frac{4(6\gamma+1)}{1-8\gamma}h^2 - \frac{16}{(1-8\gamma)l^2}, \tag{67}$$

$$-2\dot{H} + 2(2\gamma-1)\dot{h} - 3H^2 + 4(4\gamma-1)Hh + (2\gamma-1)h^2 + \frac{8\gamma+1}{4}f^2 + 12l^{-2} = 0, \tag{68}$$

$$\ddot{H} + \ddot{h} + 2(H+h)\dot{H} + (H+h)\dot{h} + hH^2 + h^2H + \frac{1}{12\alpha}h = 0, \tag{69}$$

$$\dot{H} + \dot{h} + H^2 + Hh - \frac{2\gamma}{3\alpha} + \frac{1}{12\alpha} = 0. \tag{70}$$

Eqs. (67), (68) and (70) give

$$\dot{h} = \frac{4}{8\gamma-1}H^2 - \frac{4(8\gamma-3)}{8\gamma-1}Hh + \frac{2(4\gamma+3)}{8\gamma-1}h^2 - \frac{2(16\gamma-1)}{\gamma(8\gamma-1)l^2} + \frac{8\gamma-1}{24\gamma\alpha} = 0.$$

$$\dot{H} = -\frac{8\gamma+3}{8\gamma-1}H^2 + \frac{24\gamma-11}{8\gamma-1}Hh - 2\frac{4\gamma+3}{8\gamma-1}h^2 + 2\frac{16\gamma-1}{\gamma(8\gamma-1)l^2} + \frac{(8\gamma-1)(2\gamma-1)}{24\gamma\alpha},$$

and then

$$\ddot{h} = \frac{8}{8\gamma-1}H\dot{H} - 4\frac{-3+8\gamma}{8\gamma-1}h\dot{H} - 4\frac{-3+8\gamma}{8\gamma-1}H\dot{h} + \frac{4(3+4\gamma)}{(-1+8\gamma)}h\dot{h},$$

$$\ddot{H} = -2\frac{3+8\gamma}{8\gamma-1}H\dot{H} + \frac{24\gamma-11}{8\gamma-1}h\dot{H} + \frac{24\gamma-11}{8\gamma-1}H\dot{h} - \frac{4(3+4\gamma)}{(-1+8\gamma)}h\dot{h}.$$

Substituting into (69) yields

$$h\left(\dot{H} + \dot{h}\right) + hH^2 + h^2H + \frac{1}{12\alpha}h = 0.$$

This equation and (70) lead to

$$h = 0.$$

Then (69) becomes

$$\ddot{H} + 2H\dot{H} = 0.$$

It has the solution

$$\dot{H} = -H^2 + C,$$

$$H = \sqrt{C}\frac{e^{2\sqrt{C}(t-t_0)} + 1}{e^{2\sqrt{C}(t-t_0)} - 1}.$$

The deceleration parameter is

$$q = -\frac{\dot{H}}{H^2} - 1 = -\frac{C}{H^2}.$$

When

$$C = 0$$

we have

$$-\frac{dH}{H^2} = dt,$$

and then

$$\frac{1}{H} - \frac{1}{H_0} = t - t_0.$$

B. When $\beta = -4\alpha$

In this case the gravitational Lagrangian is the square of the traceless Ricci tensor $\widetilde{R}_{\mu\nu} = R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R$ [10].

According to section III, the equation (37) gives two cases.

In the first case $f = 0$, the nonvanishing functions H and h satisfy the equations (40), (41) and (44), i.e.,

$$\begin{aligned}&\left(\dot{H} + \dot{h}\right)^2 + 2\left(\dot{H} + \dot{h}\right)H(H+h) - h(h+2H)(h+H)^2 \\ &+\frac{6\gamma+1}{4\alpha}h^2 + \frac{1}{4\alpha}H^2 + \frac{1}{2\alpha}Hh - \frac{1}{\alpha}l^{-2} = 0,\end{aligned} \tag{71}$$

$$4\left(\dot{H} + \dot{h}\right) - 8\gamma\dot{h} + 4(2\gamma+1)h^2 - 4(8\gamma-3)hH + 8H^2 - 32l^{-2} = 0, \tag{72}$$

$$\left(\ddot{H}+\ddot{h}\right)-\left(H\dot{h}+h\dot{h}+h^2H\right)-h^3+\frac{1}{8\alpha}h=0. \tag{73}$$

They can be rewritten as

$$\begin{aligned}\ddot{h} &= \left(\frac{4\gamma+3}{2\gamma}h-\frac{8\gamma-4}{2\gamma}H\right)\dot{h}-\left(\frac{8\gamma-3}{2\gamma}h-\frac{2}{\gamma}H\right)\dot{H}\\ &\quad+\frac{1}{2\gamma}h^2H+\frac{1}{2\gamma}h^3-\frac{1}{16\alpha\gamma}h,\end{aligned} \tag{74}$$

$$\begin{aligned}\ddot{H} &= \left(-\frac{2\gamma+3}{2\gamma}h+\frac{5\gamma-2}{\gamma}H\right)\dot{h}+\left(\frac{8\gamma-3}{2\gamma}h-\frac{2}{\gamma}H\right)\dot{H}\\ &\quad+\frac{2\gamma-1}{2\gamma}h^2H+\frac{2\gamma-1}{2\gamma}h^3-\frac{2\gamma-1}{16\alpha\gamma}h,\end{aligned} \tag{75}$$

and

$$\begin{aligned}&4\gamma^2\dot{h}^2-4\gamma\left((2\gamma+1)h^2-(8\gamma-2)Hh+H^2-\frac{8}{l^2}\right)\dot{h}\\ &+4\gamma(1+\gamma)h^4-8\gamma(1+4\gamma)h^3H+4\gamma(16\gamma-7)h^2H^2-16\gamma H^3h\\ &+\left(-\frac{16(2\gamma+1)}{l^2}+\frac{6\gamma+1}{4\alpha}\right)h^2+\left(32\frac{4\gamma-1}{l^2}+\frac{1}{2\alpha}\right)Hh\\ &-32\frac{H^2}{l^2}+\left(-\frac{16}{l^2}+\frac{1}{4\alpha}\right)H^2+\frac{64}{l^4}-\frac{1}{\alpha}l^{-2}\\ =\ &0.\end{aligned} \tag{76}$$

Let

$$\dot{H}=X,\dot{h}=Y.$$

We have

$$\begin{aligned}\dot{Y} &= \left(\frac{4\gamma+3}{2\gamma}h-\frac{8\gamma-4}{2\gamma}H\right)Y-\left(\frac{8\gamma-3}{2\gamma}h-\frac{2}{\gamma}H\right)X\\ &\quad+\frac{1}{2\gamma}h^2H+\frac{1}{2\gamma}h^3-\frac{1}{16\alpha\gamma}h,\end{aligned} \tag{77}$$

$$\begin{aligned}\dot{X} &= \left(-\frac{2\gamma+3}{2\gamma}h+\frac{5\gamma-2}{\gamma}H\right)Y+\left(\frac{8\gamma-3}{2\gamma}h-\frac{2}{\gamma}H\right)X\\ &\quad+\frac{2\gamma-1}{2\gamma}h^2H+\frac{2\gamma-1}{2\gamma}h^3-\frac{2\gamma-1}{16\alpha\gamma}h,\end{aligned} \tag{78}$$

and

$$\begin{aligned}&Y^2-\left(\frac{(2\gamma+1)}{\gamma}h^2-\frac{2(4\gamma-1)}{\gamma}hH+\frac{1}{\gamma}H^2-\frac{8}{\gamma l^2}\right)Y\\ &+\frac{(\gamma+1)}{\gamma}h^4-\frac{2(4\gamma+1)}{\gamma}h^3H+\frac{(16\gamma-7)}{\gamma}h^2H^2-\frac{4}{\gamma}hH^3\end{aligned}$$

$$
\begin{aligned}
&+\left(-\frac{4(2\gamma+1)}{\gamma^2 l^2}+\frac{6\gamma+1}{16\alpha\gamma^2}\right)h^2+\left(\frac{8(4\gamma-1)}{\gamma^2 l^2}+\frac{1}{8\alpha\gamma^2}\right)Hh \\
&+\left(-\frac{4}{\gamma^2 l^2}+\frac{1}{16\alpha\gamma^2}\right)H^2+\frac{16}{\gamma^2 l^4}-\frac{1}{4\alpha\gamma^2}l^{-2} \\
=&\ 0.
\end{aligned} \tag{79}
$$

The constraint equation (79) has the roots

$$
Y=-\frac{b}{2}\pm\sqrt{\left(\frac{b}{2}\right)^2-c}=Y(H,h), \tag{80}
$$

where

$$
b = -\left(\frac{(2\gamma+1)}{\gamma}h^2-\frac{2(4\gamma-1)}{\gamma}hH+\frac{1}{\gamma}H^2-\frac{8}{\gamma l^2}\right) \tag{81}
$$

$$
\begin{aligned}
c =&\ \frac{(\gamma+1)}{\gamma}h^4-\frac{2(4\gamma+1)}{\gamma}h^3H+\frac{(16\gamma-7)}{\gamma}h^2H^2-\frac{4}{\gamma^2}hH^3\gamma \\
&+\left(-\frac{4(2\gamma+1)}{\gamma^2 l^2}+\frac{6\gamma+1}{16\alpha\gamma^2}\right)h^2+\left(\frac{8(4\gamma-1)}{\gamma^2 l^2}+\frac{1}{8\alpha\gamma^2}\right)Hh \\
&+\left(-\frac{4}{\gamma^2 l^2}+\frac{1}{16\alpha\gamma^2}\right)H^2+\frac{16}{\gamma^2 l^4}-\frac{1}{4\alpha\gamma^2}l^{-2},
\end{aligned} \tag{82}
$$

So we are left with only three independent unknown functions h, H, and X, which satisfies the equations

$$
\begin{aligned}
\dot{H} &= X, \\
\dot{h} &= Y(H,h), \\
\dot{X} &= \left(-\frac{2\gamma+3}{2\gamma}h+\frac{5\gamma-2}{\gamma}H\right)Y(H,h)+\left(\frac{8\gamma-3}{2\gamma}h-\frac{2}{\gamma}H\right)X \\
&\quad+\frac{2\gamma-1}{2\gamma}h^2H+\frac{2\gamma-1}{2\gamma}h^3-\frac{2\gamma-1}{16\alpha\gamma}h.
\end{aligned} \tag{83}
$$

The critical point equations are

$$
X = 0, \tag{84}
$$

$$
Y(H,h) = 0, \tag{85}
$$

$$
\frac{2\gamma-1}{2\gamma}h^2H+\frac{2\gamma-1}{2\gamma}h^3-\frac{2\gamma-1}{16\alpha\gamma}h = 0. \tag{86}
$$

The Eq.(86) means

$$
h=0, \tag{87}
$$

or

$$
hH+h^2-\frac{1}{8\alpha}=0. \tag{88}
$$

For

$$h = 0,$$

the equations (85) has the roots

$$H = \pm\left(\frac{2}{l}\right).$$

So we have the first pair of critical points

$$X = 0, h = 0, H = \pm\left(\frac{2}{l}\right).$$

For

$$hH + h^2 - \frac{1}{8\alpha} = 0,$$

the critical point equations become

$$\begin{aligned} X &= 0, && (89)\\ H &= \left(-h + \frac{1}{8\alpha h}\right), && (90)\\ & h^6 - \left(\frac{1}{5\alpha} - \frac{3}{200\gamma\alpha} + \frac{8}{5\gamma l^2}\right)h^4 \\ & +\left(\frac{1}{100\alpha^2} + \frac{1}{320\gamma\alpha^2} + \frac{16\gamma - 1}{100\gamma^2 l^2\alpha} + \frac{16}{25\gamma^2 l^4}\right)h^2 \\ & -\frac{8\gamma - 1}{25600\alpha^3\gamma^2} - \frac{1}{400\gamma^2 l^2\alpha^2} \\ &= 0. && (91) \end{aligned}$$

The equation (91) has the roots

$$\begin{aligned} h_1^2 &= \left(-\frac{q}{2} + \sqrt{\Delta}\right)^{1/3} + \left(-\frac{q}{2} - \sqrt{\Delta}\right)^{1/3} - \frac{A}{3}, \\ h_2^2 &= \left(-\frac{q}{2} + \sqrt{\Delta}\right)^{1/3}\omega + \left(-\frac{q}{2} - \sqrt{\Delta}\right)^{1/3}\omega^2 - \frac{A}{3}, \\ h_3^2 &= \left(-\frac{q}{2} + \sqrt{\Delta}\right)^{1/3}\omega^2 + \left(-\frac{q}{2} - \sqrt{\Delta}\right)^{1/3}\omega - \frac{A}{3}, && (92) \end{aligned}$$

where

$$\begin{aligned} \Delta &= \left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3, \\ \omega &= \frac{1}{2}\left(-1 + \sqrt{3}i\right), \\ p &= B - \frac{1}{3}A^2, \\ q &= \frac{2}{27}A^3 - \frac{1}{3}AB + C, && (93) \end{aligned}$$

with

$$
\begin{aligned}
A &= -\left(\frac{1}{5\alpha}-\frac{3}{200\gamma\alpha}+\frac{8}{5\gamma l^2}\right), -\frac{A}{3}=+\frac{1}{15\alpha}-\frac{1}{200\gamma\alpha}+\frac{8}{15\gamma l^2}\\
B &= \frac{1}{100\alpha^2}+\frac{1}{320\gamma\alpha^2}+\frac{16\gamma-1}{100\gamma^2 l^2\alpha}+\frac{16}{25\gamma^2 l^4}\\
C &= -\left(\frac{8\gamma-1}{25600\alpha^3\gamma^2}+\frac{1}{400\gamma^2 l^2\alpha^2}\right).
\end{aligned}
\tag{94}
$$

The equations (89), (90) and (92) give the critical points $\{X, H, h\}$. Every one of these point corresponds to a de Sitter spacetime.

The dynamical system (83) has the Jacobian elements

$$
\begin{aligned}
\frac{\partial \dot{H}}{\partial H} &= 0, \frac{\partial \dot{H}}{\partial h}=0, \frac{\partial \dot{H}}{\partial X}=1,\\
\frac{\partial \dot{h}}{\partial H} &= \frac{\partial Y}{\partial H}, \frac{\partial \dot{h}}{\partial h}=\frac{\partial Y}{\partial h}, \frac{\partial \dot{h}}{\partial X}=0,\\
\frac{\partial \dot{X}}{\partial H} &= \frac{5\gamma-2}{\gamma}Y(H,h)+\left(-\frac{2\gamma+3}{2\gamma}h+\frac{5\gamma-2}{\gamma}H\right)\frac{\partial Y}{\partial H}-\frac{2}{\gamma}X+\frac{2\gamma-1}{2\gamma}h^2,\\
\frac{\partial \dot{X}}{\partial h} &= -\frac{2\gamma+3}{2\gamma}Y(H,h)+\left(-\frac{2\gamma+3}{2\gamma}h+\frac{5\gamma-2}{\gamma}H\right)\frac{\partial Y}{\partial h}+\frac{8\gamma-3}{2\gamma}X\\
&\quad +\frac{2\gamma-1}{\gamma}hH+3\frac{2\gamma-1}{2\gamma}h^2-\frac{2\gamma-1}{16\alpha\gamma},\\
\frac{\partial \dot{X}}{\partial X} &= \frac{8\gamma-3}{2\gamma}h-\frac{2}{\gamma}H,
\end{aligned}
\tag{95}
$$

where

$$
\begin{aligned}
\frac{\partial Y}{\partial H} &= -\frac{(4\gamma-1)}{\gamma}h+\frac{1}{\gamma}H\\
&\quad \pm\frac{1}{2\sqrt{\left(\frac{b}{2}\right)^2-c}}\left(\frac{1}{\gamma^2}h^3+\frac{3}{\gamma^2}h^2H+\frac{3}{\gamma^2}hH^2+\frac{1}{\gamma^2}H^3-\frac{1}{8\gamma^2\alpha}h-\frac{1}{8\alpha\gamma^2}H\right),
\end{aligned}
\tag{96}
$$

$$
\begin{aligned}
\frac{\partial Y}{\partial h} &= \frac{(2\gamma+1)}{\gamma}h-\frac{(4\gamma-1)}{\gamma}H\\
&\quad \pm\frac{1}{2\sqrt{\left(\frac{b}{2}\right)^2-c}}\left(\frac{1}{\gamma^2}h^3+\frac{3}{\gamma^2}h^2H+\frac{3}{\gamma^2}hH^2+\frac{1}{\gamma^2}H^3-\frac{6\gamma+1}{8\gamma^2\alpha}h-\frac{1}{8\gamma^2\alpha}H\right).
\end{aligned}
\tag{97}
$$

In order to analyze their stability we give the parameter α and γ specific values and then obtain the results:

For the critical point $X = 0, h = 0, H = 2/l$, corresponding calculation indicates it is unstable for $\alpha = \frac{1}{32}l^2$.

In the case $X = 0$, $hH + h^2 - \frac{1}{8\alpha} = 0$, we have

When

$$\alpha = \left(\frac{1}{32} l^2\right), \gamma = \left(\frac{1}{4}\right)$$

the equations (92) and (90) give

$$\begin{aligned} h_1^2 &= \frac{1.9814}{l^2}, h_1 = \pm\frac{1.4076}{l}, H_1 = \pm\frac{1.4341}{l} \\ h_2^2 &= \frac{4.4493 + 3.3485i}{l^2}, \\ h_3^2 &= \frac{4.4493 - 3.3485i}{l^2}, \end{aligned}$$

At

$$h_1 = \left(\frac{1.4076}{l}\right), H_1 = \left(\frac{1.4341}{l}\right),$$

for

$$Y = -\frac{b}{2} + \sqrt{\left(\frac{b}{2}\right)^2 - c} = 0,$$

the dynamical system (83) has the form

$$\begin{aligned} \dot{H} &= X, \\ \dot{h} &= Y(H, h), \\ \dot{X} &= -(7h + 3H)\, Y(H, h) - (7h + 3H)\, X \\ &\quad -h^2 H - h^3 + \frac{4}{l^2} h, \end{aligned}$$

with

$$\begin{aligned} Y &= 3h^2 + 2H^2 - \frac{16}{l^2} \\ &\quad +(4h^4 + 24h^2H^2 - \frac{80}{l^2}h^2 + 4H^4 - \frac{32}{l^2}H^2 \\ &\quad +\frac{128}{l^4} + 16h^3H + 16hH^3 - 64h\frac{H}{l^2})^{1/2}. \end{aligned}$$

Its Jacobian

$$\mathcal{M} = \begin{pmatrix} 0 & 0 & 1 \\ \frac{21.324}{l} & \frac{12.665}{l} & 0 \\ -\frac{303.83}{l^2} & -\frac{185.26}{l^2} & -\frac{14.288}{l} \end{pmatrix}$$

has the eigenvalues: $-0.39225/l + 11.048i/l, -0.39225/l - 11.048i/l, -0.8385/l.$ The critical point

$$h_1 = \left(\frac{1.4076}{l}\right), H_1 = \left(\frac{1.4341}{l}\right),$$

is stable, where $f = 0$.

For

$$Y = -\frac{b}{2} - \sqrt{\left(\frac{b}{2}\right)^2 - c} = -\frac{11.886}{l^2},$$

the dynamical system (83) has the Jacobian

$$\mathcal{M} = \begin{pmatrix} 0 & 0 & 1 \\ -\frac{9.8509}{l} & \frac{4.2259}{l} & 0 \\ \frac{173.12}{l^2} & \frac{17.401}{l^2} & -\frac{14.288}{l} \end{pmatrix}$$

with the eigenvalues: $-22.33/l, 6.1339/l + 1.6776i/l, 6.1339/l - 1.6776i/l.$ The critical point is unstable.

In the case $f \neq 0$, (33), (34), (35) and (39) read (in vacuum)

$$\begin{aligned} &\left(\dot{H} + \dot{h}\right)^2 + 2\left(\dot{H} + \dot{h}\right) H (H + h) \\ &-h(h + 2H)(h + H)^2 + \frac{1}{2}(h + H)^2 f^2 - \frac{1}{16} f^4 \\ &+\frac{6\gamma + 1}{4\alpha} h^2 + \frac{1}{4\alpha} H^2 + \frac{1}{2\alpha} Hh + \frac{8\gamma - 1}{16\alpha} f^2 - \frac{1}{\alpha} l^{-2} = 0, \end{aligned} \tag{98}$$

$$\left(\ddot{H} + \ddot{h}\right) - \left(H\dot{h} + h\dot{h} + h^2 H\right) - \frac{1}{2}\left(2h^3 - f\dot{f} - \frac{1}{2} h f^2\right) + \frac{1}{8\alpha} h = 0, \tag{99}$$

$$4\alpha\left(\dot{H} + \dot{h}\right) - 4\alpha Hh - \alpha\left(4h^2 - f^2\right) - 4\gamma + \frac{1}{2} = 0. \tag{100}$$

$$f^2 = 4\left(\dot{H} + \dot{h}\right) - 8\gamma\dot{h} + 4(2\gamma + 1)h^2 - 4(8\gamma - 3)hH + 8H^2 - 32l^{-2}, \tag{101}$$

These equations have the solution

$$\alpha = \left(\frac{1}{64} - \frac{1}{4}\gamma\right) l^2, \tag{102}$$

$$H^2 = \frac{32\gamma}{(16\gamma - 1) l^2}, \tag{103}$$

$$f^2 = -\frac{32(8\gamma - 1)}{l^2(16\gamma - 1)}. \tag{104}$$

For

$$\gamma > \frac{1}{16},$$

or

$$\gamma < 0,$$

we have a de Sitter solution

$$H = \frac{4}{l}\sqrt{\frac{2\gamma}{16\gamma - 1}}. \tag{105}$$

When

$$|\gamma| \gg 1,$$

we have

$$H^2 \approx \frac{2}{l^2},$$

a value speculated [14]. When

$$\frac{1}{16} < \gamma \leq \frac{1}{8}$$

f is real. In this case, it is the pseudotrace axial ingredient f of torsion that produces the effect of acceleration of cosmological expansion.

V. CONCLUSIONS

Stating from a de Sitter gauge theory a gravitational Lagrangian (13) which is identified with the Lagrangian of quadratic-curvature gravities with torsion has been constructed. The cosmological equations (33-36) for spatial flat universe have been obtained. To search for vacuum solutions of them in two specific models, the conformal model and the zero-energy (Deser-Tekin) model, the dynamical systems have been derived, some de Sitter critical points and their stability have been investigated. These points are always exact constant solutions in the context of autonomous dynamical systems and describe the asymptotic behavior. Some stable de Sitter critical points have been found. For any physical theories, to find exact mathematical solutions is an important topic. Next comes the physical interpretations of the solution thus obtained. Mathematically, de Sitter as the maximally space is undoubtedly important for any gravity theories. From observational side, recent studies illuminate that both the early universe (inflation) and the late-time universe (cosmic acceleration) can be regarded as fluctuations on a de Sitter background. So de Sitter takes a pivotal status in gravity, especially in modern cosmology.

The solutions in section IV indicate that when $f = 0$, $h \neq 0$, the cosmological equations have stable de Sitter critical points. This means that the scalar ingredient h of torsion could be considered as a ”phantom”

field, since it does not interact directly with matter; it only interacts indirectly via gravitation. In the case $f \neq 0$, $h = 0$, it is the pseudotrace axial ingredient f of torsion that produces the effect of acceleration of cosmological expansion. Therefore the spacetime in the vacuum has the structure of de Sitter spacetime with torsion including the pseudotrace axial ingredient f as well as the scalar ingredient h.

In summary, in the framework of gauge theory of gravity some cosmological models can be constructed to explain observable acceleration of cosmological expansion. The effect of acceleration of cosmological expansion in these models has the geometrical nature and is connected with geometrical structure of physical spacetime. The spacetime in the vacuum has the structure of de Sitter spacetime with torsion.

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